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# On a Partial Converse of the Montessus de Ballore Theorem in $\mathbb{C}^n$

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## 1. INTRODUCTION

The present paper has developed from a talk presented at the Summer Seminar on Complex Analysis in Trieste, Italy, July 1980. In that talk some basic results relating to Montessus de Ballore theorem in $\mathbb{C}^n$ were discussed. The main result of the present paper is a "partial converse" of the Montessus de Ballore theorem in $\mathbb{C}^n$. The motivation for this result originated from a paper of Walsh [5]. A modified version of the Montessus de Ballore theorem is also presented.

Section 2 of this paper gives notations, definitions, and some properties of rational approximants in $\mathbb{C}^n$. Section 3 deals entirely with questions of convergence leading to the main result. Throughout the paper, non-homogeneous polynomials in $\mathbb{C}^n$ are employed in the construction of the rational approximants in $\mathbb{C}^n$.

## 2. NOTATIONS, DEFINITIONS, AND SOME PROPERTIES OF $\pi_{\mu v}$

Let $\mathbb{C}$ denote the field of complex numbers and let $\mathbb{N}$ denote the set of nonnegative numbers. Let $z := (z_1,..., z_n) \in \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ $n$-times and $\hat{z}$ denote a point from $\mathbb{C}^{n-1}$ obtained by suppressing the last one of the variables of $z$ in $\mathbb{C}^n$.

Let $\sigma > 0$ and $\Delta_\sigma := \{z_j \in \mathbb{C}: |z_j| < \sigma\}$ be a disk in the $z_j$ variable centered at the origin so that $\Delta_\sigma^n := \Delta_\sigma \times \cdots \times \Delta_\sigma$ $n$ times, becomes a polydisk in $\mathbb{C}^n$. Let $\mathbb{N}^n := \mathbb{N} \times \cdots \times \mathbb{N}$ $n$ times.

We introduce the following partial ordering on $\mathbb{N}^n$. If $\alpha := (\alpha_1,..., \alpha_n)$ and $\beta := (\beta_1,..., \beta_n) \in \mathbb{N}^n$ then $0 \leqslant \alpha \leqslant \beta \Leftrightarrow 0 \leqslant \alpha_j \leqslant \beta_j$, $0 \leqslant j \leqslant n$. Next we let $E_\tau := \{\gamma \in \mathbb{N}^n: 0 \leqslant \gamma \leqslant \tau, \tau \in \mathbb{N}^n\}$. A polynomial $P_\lambda(z)$ in $\mathbb{C}^n$ can be written as



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$$P_\lambda(z) = \sum_{\gamma \in E_\lambda} g_\lambda z^\gamma \qquad \text{with} \quad g_\gamma = g_{\gamma_1 \cdots \gamma_n}$$

and $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. Such a polynomial is said to have a multiple degree at most $\lambda$ and this will be written as $m\text{-deg}(P_\lambda(z)) \leqslant \lambda$.

Let $\mathscr{R}_{\mu\nu}$ be the class of rational functions of the form $R_{\mu\nu}(z) = P_\mu(z)/Q_\nu(z)$ with $Q_\nu(0) \neq 0$, where $m\text{-deg}(P_\mu(z)) \leqslant \mu$ and $m\text{-deg}(Q_\nu(z) \leqslant \nu$, and for which $\exists \rho > 0$ and $\Delta_\rho^n$ such that $(P_\mu(z), Q_\nu(z)) = 1$, $z \in \Delta_\rho^n$ except on a subvariety of codimension $\geqslant 2$.

We now let $\mathscr{H}(U)$ denote the ring of holomorphic functions in a neighborhood $U$ of $z = 0$ and let $\mathscr{H}_*(U)$ denote the group of units of $\mathscr{H}(U)$.

Definition 1. Suppose $f \in \mathscr{H}_*(U)$. $R_{\mu\nu}(z) \in \mathscr{R}_{\mu\nu}$ is said to be a $(\mu, \nu)$-rational approximant to $f$ at $z = 0$ if

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} (Q_\nu(z) f(z) - P_\mu(z)) \bigg|_{z=0} = 0, \tag{2.1}$$

for $\lambda \in E^{\mu\nu} \subset \mathbb{N}^n$, an index interpolation set with the properties:

(i) $0 \in E^{\mu\nu}$,

(ii) $\lambda \in E^{\mu\nu} \Rightarrow \gamma \in E^{\mu\nu}, \qquad \forall 0 \leqslant \gamma \leqslant \lambda$,

(iii) $E_\mu \subset E^{\mu\nu}$,

(iv) $|E^{\mu\nu}| \leqslant \prod_{j=1}^n (\mu_j + 1) + \prod_{j=1}^n (\nu_j + 1) - 1$,

(v) each projected variable has the Padé indexing set.

Here $|E^{\mu\nu}|$ is the cardinality of $E^{\mu\nu}$, $\partial^{|\lambda|}/\partial z^\lambda \equiv \partial^{\lambda_1 + \cdots + \lambda_n}/(\partial z_1^{\lambda_1} \cdots \partial z_n^{\lambda_n})$ and the Padé indexing set is a one dimensional index set for defining unique Padé approximants in any projected variable.

*Remark.* The index set $E_\mu \subset E^{\mu\nu}$ completely covers all suffixes for indexing the coefficients of the numerator polynomial $P_\mu(z)$ of any rational approximant $R_{\mu\nu}(z)$. Thus from (2.1) we have

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} (Q_\nu(z) f(z) - P_\mu(z)) \bigg|_{z=0} = 0, \qquad \lambda \in E_\mu, \tag{2.3}$$

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} (Q_\nu(z) f(z)) \bigg|_{z=0} = 0, \qquad \lambda \in E^{\mu\nu} \backslash E_\mu. \tag{2.4}$$

From these equations one can compute a rational approximant. The latter, however, fails to produce a rational approximant with some degree of uniqueness. The question of uniqueness for rational approximants defined above is achieved by invoking a maximality condition on $E^{\mu\nu}$ (see Lutterodt [3]) and also Karlsson and Wallin [2].

DEFINITION 2. The index interpolation set $E^{\mu\nu}$ is said to be *maximal* if and only if

$$|E^{\mu\nu}| \geqslant \prod_{j=1}^{n} (\mu_j + 1) + \prod_{j=1}^{n} (\nu_j + 1) - 1.$$

For the rest of this paper we shall assume that for each pair $(\mu, \nu)$, $\nu \leqslant \mu$ and that the rational approximants discussed, are *unisolvent*; this means that they are $(\mu, \nu)$-rational approximants for which $E^{\mu\nu}$ is maximal and for which a certain determinantal or rank condition is satisfied (see Lutterodt [3]). We shall normalize the denominator polynomial of the $(\mu, \nu)$-rational approximant, dividing the latter top and bottom through by the modulus of the former's largest coefficient. This operation leaves the unisolvent $(\mu, \nu)$-rational approximant invariant. We denote the latter by

$$\pi_{\mu\nu}(z) = \tilde{P}_{\mu\nu}(z)/\tilde{Q}_{\mu\nu}(z). \tag{2.4}$$

We shall require that if $\tilde{P}_{\mu\nu}(z)$, $\tilde{Q}_{\mu\nu}(z)$ have any uniform limits on compact subset as $\mu$ tends to infinity, then these uniform limits must remain relatively prime except on some subvariety of codimension $\geqslant 2$. Some properties of $(\mu, \nu)$ and $(\mu, \mu)$ unisolvent rational approximants are highlighted by the following theorems.

THEOREM 2.1. *Suppose $f \in \mathscr{H}_*(U)$. Let $\pi_{\mu\nu}(z)$ be unisolvent $(\mu, \nu)$-rational approximant to $f$ in $U$. Then*

(i) *$\pi_{\mu\nu}(z)$ is invertible in $V \subset U$, $V$ being 0-neighborhood.*

(ii) *$\pi_{\mu\nu}^{-1}$ is a $(\nu, \mu)$-rational approximant to $f^{-1} \in \mathscr{H}_*(U)$.*

THEOREM 2.2. *Suppose $f \in \mathscr{H}_*(U)$. Let $\pi_{\mu\mu}(z)$ be a diagonal unisolvent rational approximant to $f$ on $U$. Suppose $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Then $(a\tilde{Q}_{\mu\mu}(z) + b\tilde{P}_{\mu\mu}(z))/(c\tilde{Q}_{\mu\mu}(z) + d\tilde{P}_{\mu\mu}(z))$ is a diagonal unisolvent rational approximant to the meromorphic function $(a + bf(z))/(c + df(z))$, where $c + df(z) \neq 0$.*

The next theorem also concerns diagonal unisolvent rational approximants and their change under the biholomorphic mapping $\phi: U \to V$ defined by $\phi(z) = (a_1 z_1/(c_1 z_1 + d_1),..., a_n z_n/(c_n z_n + d_n))$ with $\phi(0) = 0$, where $V$ is another neighborhood of $z = 0$ and $a_i$, $c_i$, and $d_i \in \mathbb{C}$, $a_i d_i \neq 0$, $c_i z_i + d_i \neq 0$, $i = 1,..., n$.

THEOREM 2.3. *Suppose $f \in \mathscr{H}_*(U)$. Let $\phi: U \to V$ be as defined above. Let $\pi_{\mu\mu}(z)$ be a diagonal unisolvent rational approximant to $f$ in $U$. Then $\pi_{\mu\mu} \circ \phi$ is a diagonal unisolvent rational approximant to $f \circ \phi$ in $V$.*

Proofs for the above theorems were given in Lutterodt [7] even though the statements of the same theorems in the present paper differ from those of the 1976 paper.

## 3. Convergence

The main result of this section is a "partial converse" of the $\mathbb{C}^n$-version of the Montessus de Ballore theorem. In this section, we relax the definition of rational approximants for $f \in \mathscr{H}_*(U)$ to include $f \in \mathscr{H}(U)$ with possibly $f(0)=0$ (see Lutterodt [3]).

Let $\Delta_\rho^n$ be a polydisk domain centered at the origin and let $U \subset \Delta^n$ be an 0-neighborhood. We shall denote by $\mathfrak{M}^1(\Delta_\rho^n)$ the class of functions on $\mathbb{C}^n$ that are holomorphic on $U$ and meromorphic in $\Delta_\rho^n$ with polar sets of finitely many sections defined by a polynomial on $\Delta_\rho^n$ having minimal $m$-degree. Thus if $f \in \mathfrak{M}^1(\Delta_\rho^n)$ and has a polar set defined by $q_\nu(z)=0$ with minimal $m$-$\deg(q_\nu(z))=\nu$, then we shall write $Z_\nu(f^{-1})=\{z \in \mathbb{C}^n : q_\nu(z)=0\}$.

Theorem 3.1. *Let $\nu \in \mathbb{N}^n$ and $\rho > 0$ be fixed. Suppose $f \in \mathfrak{M}^1(\Delta_\rho^n)$ and $Z_\nu(f^{-1})$ is the pole set of $f$ defined on $\Delta_\rho^n$ by $q_\nu(z)=0$ with minimal $m$-$\deg(q_\nu(z))=\nu$ so that $Z_\nu(f^{-1}) \cap \Delta_\rho^n \neq \varnothing$ and $q_\nu f \in C(\bar{\Delta}_\rho^n)$.*

Let $\pi_{\mu\nu}(z)$ be a unisolvent $(\mu,\nu)$-rational approximant to $f$ at 0, where the polar set of $\pi_{\mu\nu}(z)$ denoted by $Q_{\mu\nu}^{-1}(0)=\{z \in \mathbb{C}^n : \tilde{Q}_{\mu\nu}(z)=0\}$ is such that for sufficiently large $\mu$, $Q_{\mu\nu}^{-1}(0) \cap \Delta_\rho^n \neq \varnothing$. Let $\mu' = \min_{1 \leqslant i \leqslant n}(\mu_i)$. Then as $\mu' \to \infty$,

(i) $Q_{\mu\nu}^{-1}(0) \cap \Delta_\rho^n \to Z_\nu(f^{-1}) \cap \Delta_\rho^n$,

(ii) $\pi_{\mu\nu}(z) \to f(z)$ almost uniformly on compact subsets of $\Delta_\rho^n$.

Next we state the partial converse of Theorem 3.1.

Theorem 3.2. *Let $\rho > 0$ and $\nu \in \mathbb{N}^n$ be fixed. Let $U$ be an 0-neighborhood. Suppose $f \in \mathscr{H}(U)$ and suppose $\pi_{\mu\nu}(z)$ is a unisolvent $(\mu,\nu)$-rational approximant to $f(z)$ at 0. Let $\Delta_\rho^n$ be a polydisk in $\mathbb{C}^n$ such that $U \subset \Delta_\rho^n$ and for all $\mu > \mu_0$ ($\mu_0$ being chosen in $\mathbb{N}^n$),*

$$Q_{\mu\nu}^{-1}(0) \cap \Delta_\rho^n \neq \varnothing.$$

For each fixed $\hat{a} \in \Delta_\rho^{n-1} \subset \mathbb{C}^{n-1}$, suppose the poles of $\pi_{\mu\nu}(\hat{a}, z_n)$ as a rational function of a single variable in $z_n$, are uniformly bounded with respect to $\hat{a}$ and $\mu$ in $|z_n| < \rho$. Suppose each subsequence of $\{\pi_{\mu\nu}(\hat{a}, z_n)\}_\mu$ of the respective unisolvent $(\mu,\nu)$-rational approximants converges uniformly to $f(\hat{a}, z_n)$ on every compact subset of $\{|z_n| < \rho\}$ not containing limit points of poles of that subsequence. Then $f \in \mathfrak{M}^1(\Delta_\rho^n)$ with at most $\nu_n$ codimension one simple polar sections.

*Proof of Theorem* 3.1. We shall assume without loss of generality that $q_\nu(z)$ is normalized in the same way as $\tilde{Q}_{\mu\nu}(z)$. Let $H_{\mu\nu}(z)$ be defined by

$$H_{\mu\nu}(z) = \tilde{Q}_{\mu\nu}(z)\, q_\nu(z)\, f(z) - q_\nu(z)\, \tilde{P}_{\mu\nu}(z). \tag{3.1}$$

Then from the hypothesis of the theorem $H_{\mu\nu}(z) \in C(\bar{\Delta}_\rho^n)$ and $H_{\mu\nu}(z) \in \mathscr{H}(\Delta_\rho^n)$; thus by the Cauchy's integral formula,

$$H_{\mu\nu}(z) = \frac{1}{(2\pi i)^n} \int_T \frac{H_{\mu\nu}(t)}{\prod_{j=1}^n (t_j - z_j)}\, dt_1 \cdots dt_n, \tag{3.2}$$

where $T$ is the distinguished boundary of $\Delta_\rho^n$. Now

$$\prod_{j=1}^{n} (t_j - z_j)^{-1} = \sum_{\lambda \in \mathbb{N}^n} \frac{z^\lambda}{t^{\lambda+1}} \tag{3.3}$$

is uniformly convergent on compact subsets of $\Delta_\rho^n$; thus an interchange of $\sum$ and $\int$ yields

$$H_{\mu\nu}(z) = \sum_{\lambda \in \mathbb{N}^n} h_{\mu\nu\lambda} z^\lambda, \tag{3.4}$$

with

$$h_{\mu\nu\lambda} = \frac{1}{(2\pi i)^n} \int_T \frac{H_{\mu\nu}(t)}{t^{\lambda+1}}\, dt_1 \cdots dt_n, \tag{3.5}$$

for $\nu$ fixed, for all $\lambda$, and all $\mu$ such that $\nu \leqslant \mu$. But from (2.2) and (2.3) using generalized Leibnitz rule we get

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} [q_\nu(z)(\tilde{Q}_{\mu\nu}(z)\, f(z) - \tilde{P}_{\mu\nu}(z))] \Big|_{z=0} = 0, \qquad \lambda \in E_\mu,$$

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} [q_\nu(z)(\tilde{Q}_{\mu\nu}(z)\, f(z))] \Big|_{z=0} = 0, \qquad \lambda \in E^{\mu\nu} \backslash E_\mu.$$

Thus in the expansion (3.4) we find that,

$$H_{\mu\nu}(z) = \sum_{\lambda \in \mathbb{N}^n \backslash E^{\mu\nu}} h_{\mu\nu\lambda} z^\lambda, \tag{3.4a}$$

with

$$h_{\mu\nu\lambda} = \frac{1}{(2\pi i)^n} \int_T \frac{\tilde{Q}_{\mu\nu}(t)\, q_\nu(t)\, f(t)}{t^{\lambda+1}}\, dt_1 \cdots dt_n. \tag{3.5a}$$

Since by design $\tilde{Q}_{\mu\nu}(z)$ is locally bounded in $\mathbb{C}^n$ and $q_\nu(z) f(z) \in C(\bar{\Delta}^n_\rho)$, $\exists$ a constant $M = M(\rho)$ such that $|\tilde{Q}_{\mu\nu}(t) q_\nu(t) f(t)| \leqslant M$, $\forall t \in T$ yielding a Cauchy-type inequality from (3.5a) as

$$|h_{\mu\nu\lambda}| \leqslant \frac{M}{\rho^{|\lambda|}}, \qquad \rho > 0, \tag{3.6}$$

where $|\lambda| = \sum_{j=1}^n \lambda_j$ and r.h.s. of (3.6) is independent of $\mu$. Thus combining (3.4a), and (3.6), noting $E_\mu \subset E^{\mu\nu}$, we get

$$|H_{\mu\nu}(z)| \leqslant M \sum_{\lambda \in \mathbb{N}^n \setminus E_\mu} \frac{|z^\lambda|}{\rho^{|\lambda|}}. \tag{3.7}$$

The r.h.s. of (3.7) being a tail of a geometric series in $\mathbb{R}^n$ tends to zero as $\mu' \to \infty$. Hence $H_{\mu\nu}(z) \to 0$ pointwise in $\Delta^n_\rho$ and uniformly on compact subsets of $\Delta^n_\rho$ as $\mu' \to \infty$.

Now the sequence $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$ is locally bounded on $\Delta^n_\rho$ and therefore $\{\tilde{Q}_{\mu\nu}(z)\}_\mu \subset \mathscr{F}$, a normal family. Thus it has uniformly convergent subsequence on compact subsets of $\Delta^n_\rho$. This fact together with the uniform null convergence of $H_{\mu\nu}(z)$ in compact subsets of $\Delta^n_\rho$, induces a uniform convergence of a similar subsequence of $\{\tilde{P}_{\mu\nu}(z)\}_\mu$ on compact subsets of $\Delta^n_\rho$. If we let $Q_\nu(z)$ and $P(z)$ be the uniform limits to the convergent subsequences of $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$ and $\{\tilde{P}_{\mu\nu}(z)\}_\mu$, respectively, then in the limit as $\mu' \to \infty$, we obtain from $H_{\mu\nu}(z) \to 0$,

$$Q_\nu(z) q_\nu(z) f(z) = q_\nu(z) P(z), \tag{3.8}$$

for $z \in \Delta^n_\rho$. Suppose $a \in Z_\nu(f^{-1}) \cap \Delta^n_\rho$, then $q_\nu(a) = 0 \Rightarrow Q_\nu(a) q_\nu(a) f(a) = 0$. But $q_\nu(z) f(z) \neq 0$ except for a subvariety of codimension $\geqslant 2$, therefore $Q_\nu(a) = 0 \Rightarrow a \in Q_\nu^{-1}(0) \cap \Delta^n_\rho$, where $Q_\nu^{-1}(0)$ is the zero set of $Q_\nu(z)$. Thus

$$Z_\nu(f^{-1}) \cap \Delta^n_\rho \subset Q_\nu^{-1}(0) \cap \Delta^n_\rho.$$

Conversely, $a \in Q_\nu^{-1}(0) \cap \Delta^n_\rho \Rightarrow Q_\nu(a) = 0 \Rightarrow q_\nu(a) P(a) = 0$ from (3.8). But $(Q_\nu(z), P(z)) = 1$ except for a subvariety of codimension $\geqslant 2$. Thus $q_\nu(a) = 0 \Rightarrow a \in Z_\nu(f^{-1}) \cap \Delta^n_\rho$, i.e., $Z_\nu(f^{-1}) \cap \Delta^n_\rho \supset Q_\nu^{-1}(0) \cap \Delta^n_\rho$. Hence

$$Z_\nu(f^{-1}) \cap \Delta^n_\rho = Q_\nu^{-1}(0) \cap \Delta^n_\rho. \tag{3.9}$$

*Claim.* Every subsequence of $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$ and, consequently, every subsequence of $\{\tilde{P}_{\mu\nu}(z)\}_\mu$ contain subsequences that are uniformly convergent to $Q_\nu(z)$ and $P(z)$, respectively, on compact subsets of $\Delta^n_\rho$.

The claim together with Vitali's theorem will ensure the uniform convergence of the full sequence $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$ and $\{\tilde{P}_{\mu\nu}(z)\}_\mu$ to $Q_\nu(z)$ and $P(z)$, respectively, on compact subsets of $\Delta^n_\rho$.

To see the claim, take any subsequence of $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$ and its induced subsequence of $\{\tilde{P}_{\mu\nu}(z)\}_\mu$ and select, respectively, two new subsequences from the two originally taken that converge uniformly to say, $S_\nu(z)$ and $W(z)$, respectively, on compact subsets of $\Delta_\rho^n$. Then these satisfy (3.8) and consequently, we obtain an analogue of (3.9) with $S_\nu^{-1}(0)\cap\Delta_\rho^n = Z_\nu(f^{-1})\cap\Delta_\rho^n$ so that on $Z_\nu(f^{-1})\cap\Delta_\rho^n$, $S_\nu(z)=q_\nu(z)$. This holds for any uniform limits of uniformly convergent subsequences of $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$. We call the common uniform limit $Q_\nu(z)=q_\nu(z)$ on $Z_\nu(f^{-1})\cap\Delta_\rho^n$ and so on $\Delta_\rho^n$. To show that convergent subsequences of $\{\tilde{P}_{\mu\nu}(z)\}_\mu$ also have common uniform limit, suppose there are two $W_1(z)$ and $W_2(z)$. Then these satisfy, from (3.8),

$$q_\nu(z)\,W_1(z)=Q_\nu(z)\,q_\nu(z)\,f(z)=q_\nu(z)\,W_2(z).$$

Thus we get either $q_\nu(z)=0$ or $W_1(z)=W_2(z)$ except on a subvariety of codimension $\geqslant 2$, so that when $q_\nu(z)\neq 0$ in $\Delta_\rho^n$ we call the common uniform limit of uniformly convergent subsequences of $\{\tilde{P}_{\mu\nu}(z)\}_\mu$, $P(z)$. This establishes the claim. Thus (3.9) holds for the full sequences $\{\tilde{Q}_{\mu\nu}(z)\}_\mu$ and $\{\tilde{P}_{\mu\nu}(z)\}_\mu$; therefore as $\mu'\to\infty$,

$$Q_{\mu\nu}^{-1}(0)\cap\Delta_\rho^n\to Z_\nu(f^{-1})\cap\Delta_\rho^n.$$

This concludes (i).

We now turn to the proof of (ii). From (3.7),

$$|\tilde{Q}_{\mu\nu}(z)\,q_\nu(z)\,f(z)-q(z)\,\tilde{P}_{\mu\nu}(z)|\leqslant M\sum_{\lambda\in\mathbb{N}^n\setminus E_\mu}\frac{|z^\lambda|}{\rho^{|\lambda|}}.$$

Let $K$ be any compact subsets of $\Delta_\rho^n\setminus Z_\nu(f^{-1})$ and let $\rho'>0$ be chosen so that $0<\rho'<\rho\Rightarrow\Delta_{\rho'}^n\subsetneq\Delta_\rho^n$ and $K\subset\Delta_{\rho'}^n$. Then on $K$, $q_\nu(z)\neq 0$ and therefore for $\mu'$ sufficiently large, we get $\tilde{Q}_{\mu\nu}(z)\neq 0$. Thus we can find $\delta>0$ such that for $\mu'$ sufficiently large,

$$|\tilde{Q}_{\mu\nu}(z)|>\delta \qquad\text{and}\qquad |q_\nu(z)|>\delta,\quad z\in K.$$

In terms of sup-norm, we obtain for $\mu'$ sufficiently large, that

$$\|f(z)-\pi_{\mu\nu}(z)\|_K\leqslant\frac{M}{\delta^2}\sum_{\lambda\in\mathbb{N}^n\setminus E_\mu}\left(\frac{\rho'}{\rho}\right)^{|\lambda|}. \tag{3.10}$$

But $\sum_{\lambda\in\mathbb{N}^n\setminus E_\mu}(\rho'/\rho)^{|\lambda|}\leqslant n(\rho'/\rho)^{\mu'+1}/(1-(\rho'/\rho))^n$ (see Appendix). By letting $C_1$ be dependent on $M$, $n$, $\delta$, $\rho$, $\rho'$, we deduce from (3.10) that on $K$,

$$\|f(z)-\pi_{\mu\nu}(z)\|_K\leqslant C_1\left(\frac{\rho'}{\rho}\right)^{\mu'+1}, \tag{3.12}$$

so that,

$$\overline{\lim_{\mu'\to\infty}} \|f(z)-\pi_{\mu\nu}(z)\|_K^{1/\mu'} \leqslant \frac{\rho'}{\rho} < 1.$$

Now given $\varepsilon > 0$, and $\mu'$ sufficiently large, the set,

$$\{z \in \Delta_\rho^n: \|f(z)-\pi_{\mu\nu}(z)\|_{\Delta_\rho^n}^{1/\mu'} \geqslant \varepsilon\} = (Z_\nu(f^{-1}) \cup Q_{\mu\nu}^{-1}(0)) \cap \Delta_\rho^n.$$

But for sufficiently large $\mu'$, $Q_{\mu\nu}^{-1}(0)$ is close to $Z_\nu(f^{-1})$ in $\Delta_\rho^n$ by part (i) and thus it is sufficient to show that $Z_\nu(f^{-1})$ is a $2n$-dimensional Lebesgue measure zero set to clinch the result. But this result follows from Lemma 10 in Gunning and Rossi [6, p. 9].

*Proof of Theorem* 3.2. In the sequel, we shall assume, for simplicity, that all codimension one polar sections (hyperplanes) are simple in the sense that there are no "branching points." This effectively means that the unique factorization of the denominator polynomial $\tilde{Q}_{\mu\nu}(z)$ as a pseudopolynomial $\tilde{Q}_{\mu\hat{\nu}\nu_n}(\hat{z}, z_n)$ in $z_n$ yields only simple irreducible factors up to a unit factor.

According to the hypothesis of the theorem, for each fixed $\hat{a} \in \Delta_\rho^{n-1} \subset \mathbb{C}^{n-1}$, the poles of the sequence $\{\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n)\}_\mu$ are uniformly bounded w.r.t. $\hat{a}$ and $\mu$ in $|z_n| < \rho$. Thus the Bolzano–Weierstrass theorem, limit points of poles exist in $|z_n| \leqslant \rho$. Such limit points of poles in $|z_n| \leqslant \rho$ will be called *polar limits*. In order to characterize the nonspurious polar limits as a means of distinguishing them from the spurious ones, we introduce presently some criteria of admissibility.

First we form a descending chain of subsequences from the sequence $\{\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n)\}_\mu$ of rational approximants,

$$\{\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n)\}_\mu \supset \{\pi_{\bar{\mu}_k\hat{\nu}\nu_n}(\hat{a}, z_n)\}_k \supset \cdots \tag{3.14}$$

where $\mu_0 < \mu < (\mu_{1k},..., \mu_{nk}) = \bar{\mu}_k < \bar{\mu}_{k+1} = (\mu_{1k+1},..., \mu_{nk+1})$, $k = 1, 2,...,$ and each denominator pseudopolynomial $Q_{\bar{\mu}_k\hat{\nu}\nu_n}(\hat{a}, z_n)$ has degree at most $\nu_n$. Each succeeding subsequence must possess the *admissible polar limits* of the preceding subsequences in the chain as well as unveil a new *admissible polar limit* if the latter is not exhausted. By an *admissible polar limit* we mean a *uniform polar limit* with respect to compact subsets in $\Delta_\rho^{n-1}$, which

(i) lies in $|z_n| < \rho$, and

(ii) belongs to every subsequence in the descending chain.

Under the conditions of allowing only admissible polar limits, the descending chain (3.14) terminates in accordance with the well-known descending chain condition. This is because the number of admissible polar limits (associated with the nonspurous polar limits) are directly constrained by the degree for the denominator pseudopolynomials which is at most $\nu_n$.

We now let $\phi_j(\hat{a})$, $1 \leqslant j \leqslant m$, $m \leqslant \nu_n$ be the nonspurious limits distinct in $|z_n| < \rho$ which are, in fact, the uniform limit of the following poles of $\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n)$, say $\psi_{\mu_j}(\hat{a})$, $1 \leqslant j \leqslant m$. Note that under the criteria for admissibility, $\psi_{\mu_j}(\hat{z}) \to \phi_j(\hat{z})$ uniformly on compact subsets of $\Delta_\rho^{n-1}$ as $\mu' \to \infty$. Now since every subsequence of $\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n)$ converges uniformly to $f(\hat{a}, z_n)$ on compact subsets of $\{z_n : |z_n| < \rho\} \backslash \{\phi_1(\hat{a}), \ldots, \phi_m(\hat{a})\}$, we must therefore have as $\mu' \to \infty$,

$$\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n) \prod_{j=1}^{m} (z_n - \psi_{\mu_j}(\hat{a})) \to f(\hat{a}, z_n) \prod_{j=1}^{m} (z_n - \phi_j(\hat{a})),$$

uniformly on compact subsets of $|z_n| < \rho$. If we now define $F(z) = F(\hat{z}, z_n)$ by

$$F(\hat{z}, z_n) = f(\hat{z}, z_n) \prod_{j=1}^{m} (z_n - \phi_j(\hat{z})),$$

then we have to show that $F(z)$ is holomorphic in $\Delta_\rho^n$ in order to clinch the result.

Now since $\psi_{\mu_j}(\hat{z})$ are holomorphic by definition in $\Delta_\rho^{n-1}$, their uniform admissible limit $\phi_j(\hat{z})$ on compact subsets of $\Delta_\rho^{n-1}$ (with values in $|z_n| < \rho$) must be holomorphic also. Using Hensel's lemma (Grauert and Fritzsche [1]) one retrieves from the polynomial factorization at the point $(\hat{a}, z_n) \in \Delta_\rho^n$ given by $\prod_{j=1}^m (z_n - \phi_j(\hat{a}))$ the same decomposition into factors for the pseudopolynomial in $\Delta_\rho^n$ given by $\prod_{j=1}^m (z_n - \phi_j(\hat{z}))$. Since the $\phi_j(\hat{z})$ are holomorphic the latter product is holomorphic in $\mathbb{C}^n$ and has zero sections given by $\bigcup_{j=1}^m \{z_n = \phi_j(\hat{z})\}$. At each point $(\hat{a}, z_n)$ in $\Delta_\rho^n \backslash \{\phi_1(\hat{a}), \ldots, \phi_m(\hat{a})\}$, $f(\hat{a}, z_n)$ is the uniform limit of $\pi_{\mu\hat{\nu}\nu_n}(\hat{a}, z_n)$ on compact subsets and, moreover, $f \in \mathscr{H}(U)$, $U \subset \Delta_\rho^n$, an 0-neighborhood, but $U \cap \bigcup_{j=1}^m \{z_n = \phi_j(\hat{z})\} = \varnothing$; it then follows from the connectedness of $\Delta_\rho^n \backslash \bigcup_{j=1}^m \{z_n = \phi_j(\hat{z})\}$, that $f(z)$ has a holomorphic continuation from $U$ into $\Delta_\rho^n \backslash \bigcup_{j=1}^m \{z_n = \phi_j(\hat{z})\}$. Hence $F(z) = f(z) \prod_{j=1}^m (z_n - \phi_j(\hat{z})) \in \mathscr{H}(\Delta_\rho^n) \Rightarrow f \in \mathfrak{M}^1(\Delta_\rho^n)$ with at most $\nu_n$ codimension one polar sections. This concludes the proof.

## Appendix

**Lemma.** *For $0 < \rho' < \rho$ and $\mu' = \min_{1 \leqslant j \leqslant n}(\mu_j)$,*

$$\sum_{\lambda \in \mathbb{N}^n \backslash E_\mu} \left(\frac{\rho'}{\rho}\right)^{|\lambda|} \leqslant n \left(\frac{\rho'}{\rho}\right)^{\mu'+1} \Bigg/ \left(1 - \frac{\rho'}{\rho}\right)^n.$$

*Proof.* Recall that $E_\mu = \{\gamma \in \mathbb{N}^n : 0 \leqslant \gamma \leqslant \mu\}$. Then

$$\sum_{\lambda \in \mathbb{N}^n \setminus E_\mu} \left(\frac{\rho'}{\rho}\right)^{|\lambda|} \leqslant \sum_{j=1}^{n} \sum_{\lambda_j = \mu_j + 1}^{\infty} \left(\frac{\rho'}{\rho}\right)^{\lambda_j} \prod_{\substack{k=1 \\ j \neq k}}^{n} \left(\lambda \sum_{k=0}^{\infty} \left(\frac{\rho'}{\rho}\right)^{\lambda_k}\right)$$

$$= \sum_{j=1}^{n} \left(\frac{\rho'}{\rho}\right)^{\mu_j + 1} \Big/ \left(1 - \frac{\rho'}{\rho}\right)^n$$

$$\leqslant n \left(\frac{\rho'}{\rho}\right)^{\mu' + 1} \Big/ \left(1 - \frac{\rho'}{\rho}\right)^n,$$

where $\mu' = \min_{1 \leqslant j \leqslant n}(\mu_j)$.

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